UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1202**

ASSESSMENT PATTERN : **MATH1202B**

MODULE NAME : **Algebra 2**

DATE : **09-May-12**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2011/12-MATH1202B-001-EXAM-192

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the h, k-lemma. Find h, k such that $18h + 25k = 1$. Hence find an integer x such that $x \equiv 4 \pmod{18}$ and $x \equiv 6 \pmod{25}$.

   (b) Let p be a prime in $\mathbb{Z}$, and $a, b \in \mathbb{Z}$. Prove that if p divides ab then p divides a or p divides b.

   (c) Prove that there are infinitely many primes of the form $6n + 5$.

2. Give the definition of a group G, defining the terms you use. In each of the following cases, determine if G is a group, justifying your answer:

   (i) $G = \{x \in \mathbb{Q} : x \neq 2\}$, $g \star h = gh - 2g - 2h + 6$,

   (ii) $G = \{x \in \mathbb{R} : x \geq 0\}$: $g \star h = |h - g|$,

   (iii) $G = \{x \in \mathbb{R} : x \geq 0\}$: $g \star h = g + h$.

3. (a) State and prove Lagrange's Theorem.

   (b) Prove that in any finite group the order of an element divides the order of the group. Deduce that if p is a prime and $\mathbb{Z}_p^*$ the group of non-zero integers (mod p) under multiplication, then $\overline{a}^{p-1} = \overline{1}$ for all $\overline{a} \in \mathbb{Z}_p^*$.

   (c) Find $\overline{3}^{1203}$ in $\mathbb{Z}_{13}^*$.

   (d) Solve $\overline{x}^5 = \overline{7}$ in $\mathbb{Z}_{13}^*$.

4. (a) Let $A = (a_{ij})$ be an $n \times n$ matrix. Give the definition of $\det(A)$. Prove that if A is lower triangular, then $\det A = a_{11}a_{22}\ldots a_{nn}$.

(b) State, without proof, the effect on the determinant of each type of elementary row (or column) operation.

(c) Let A be the $n \times n$ matrix defined by $a_{ii} = x$, $a_{ij} = 1$ for $i \neq j$. Find $\det A$, justifying your answer. For which values of x is the matrix A invertible?

5. Let $A = \begin{pmatrix} 7 & -12 \\ 2 & -3 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Solve the system of differential equations

$$\begin{aligned} dx/dt &= 7x - 12y \\ dy/dt &= 2x - 3y \end{aligned}$$

given that $x(0) = 0$, $y(0) = 1$.

6. Throughout this question A denotes an $n \times n$ matrix over $\mathbb{C}$.

(a) State the basic criterion for A to be diagonalisable.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of A and let E_{λ_i} be the eigenspace associated to λ_i. Prove that the sum $\sum_{i=1}^{r} E_{\lambda_i}$ is direct.

(c) If $A^2 = A$ show that the only possible eigenvalues of A are 0 and 1 and that $\mathbb{C}^n = E_0 + E_1$.

(d) Deduce that if $A^2 = A$ then A is diagonalisable.